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PASCH'S AXIOM AND PROJECTIVE SPACES

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Let π be a generalized projective geometry and $i \in Z^+$ such that some i-dimensional subspace of π contains finitely many $(i-1)$-dimensional subspaces. We give a characterization of the incidence structure formed by the $(i-1)$-dimensional and i-dimensional subspaces of π (where the incidence relation is set inclusion). The specialization of this characterization to projective geometries follows. Let ρ be a connected incidence structure with more than one point and line, and let two lines have at most one point in common. If (i) both ρ and the dual of ρ satisfy Pasch's Axiom, (ii) for all points x and lines m of ρ not containing x, the number of points of m collinear with x is not 1 or 2, (iii) some line has finitely many points, then ρ is the incidence structure having the $(i-1)$-dimensional and i-dimensional subspaces (for some finite i) of some projective geometry of finite order as its points and lines respectively.

1. Introduction

We write an incidence structure π as a triple (P, L, I) where P and L are nonempty sets (whose elements are called points and lines respectively) and I is a relation. The *dual* of $\pi = (P, L, I)$ is the incidence structure $\pi^* = (L, P, I^*)$ where for $x \in P$ and $m \in L$, x and m are incident in either both or neither of π, π^*. The *adjacency graph* of π is the graph having vertex set P and two points x and y adjacent if some line is incident with both. (Then x and y will also be called *adjacent* points of π.) The *line graph* of π is the adjacency graph of π^*, so it has vertex set L, and two lines are adjacent in the line graph iff some point is common to both. The *bipartite graph* of π is the graph with vertex set $P \cup L$, and two vertices adjacent if they are incident in π. We say that π is *connected* if its bipartite graph is connected.

We call an incidence structure π *semilinear* if two points are contained in at most one common line. Clearly π is semilinear iff its dual also is. π is *linear* iff each pair of lines is connected in exactly one line. Where π is semilinear and lines m and n intersect in a point x, a line h is a *transversal* of m and n if h intersects m and n but does not contain x. Where π is semilinear and x and y are adjacent points of π we denote by xy the unique line containing x and y; likewise where m and n are intersecting lines in π we denote by $m \cap n$ the point incident with both.

We say a semilinear incidence structure satisfies *Pasch's Axiom* if for every pair of intersecting lines m and n, all transversals of m and n intersect. The dual of π satisfies Pasch's Axiom iff π has the property that for any adjacent points x and y, if u and w are points adjacent to both x and y but are not on xy, then u is adjacent to w.

For any point x of an incidence structure $\pi=(P, L, I)$ we let $r(x)$ be the cardinality of the set of lines containing x, and $r(\pi)=\min\{r(x): x\in P\}$. For any nonincident point x and line m, $t(x, m)$ is defined to be the number of points which are incident with m and adjacent to x.

Let V be a vector space. The incidence structure whose points and lines are the 1- and 2-dimensional subspace of V has a classical characterization as the points and lines of a Desarguesian projective geometry. Where V has finite dimension d over $\mathrm{GF}(q)$ Dembowski and Wagner [2], and Kantor [3], gave characterizations of the incidence structure formed by the 1- and $(d-1)$-dimensional subspaces of V. Ray-Chaudhuri and Sprague [5] characterized the incidence structure formed by the i- and $(i+1)$-dimensional subspaces of V, where V is finite dimensional over $\mathrm{GF}(q)$, by certain parametric conditions. In Theorem 1 of this paper we give a geometric characterization of this incidence structure. Theorem 2 is the corresponding characterization for the incidence structure formed by the i- and $(i+1)$-subsets of a set X, and Theorem 3 (which is stated in Section 2) is the generalization of these two results to generalized projective geometries.

Theorem 1. *Let $\pi=(P, L, I)$ be a connected semilinear incidence structure, such that*

(h1) *π satisfies Pasch's Axiom;*

(h2) *the dual of π satisfies Pasch's Axiom;*

(h3') *for any point x and line m not containing x, $t(x, m)\neq 1, 2$;*

(h4) *some line has finitely many points.*

Then one of the following occurs.

(i) $|P|=1$ *or* $|L|=1$.

(ii) *π is a projective plane.*

(iii) *For some prime power q, some vector space V of dimension d ($d\geq 4$ is a cardinal number) over $\mathrm{GF}(q)$, and some finite integer i, $0<i<d-1$, $\pi\cong(W_i, W_{i+1}, \subseteq)$ where W_j is the set of j-dimensional subspaces of V ($j=i, i+1$).*

Let i be a nonnegative integer. For any set X we represent the set of i-subsets of X by $\mathcal{P}_i(X)$.

Theorem 2. *Let $\pi=(P, L, I)$ be a connected semilinear incidence structure satisfying* (h1), (h2), (h4), *and*

(h3") *for any point x and line m not containing x, $t(x, m)\in\{0, 2\}$.*

Then for some set X, $\pi\cong(\mathcal{P}_i(X), \mathcal{P}_{i+1}(X), \subseteq)$ where i is a finite integer $0\leq i<|X|$.

Theorems 1 and 2 are useful in characterizing certain higher dimensional partial geometries, called partial geometric lattices by Bose.

2. Generalized projective geometries

A *generalized projective geometry* is a linear incidence structure satisfying Pasch's Axiom, and such that each line contains at least two points. A *projective geometry* is a generalized projective geometry such that each line has at least three points. A *subspace* of a generalized projective geometry $\rho = (P, L, I)$ is a set S of points such that for all lines m if S contains at least two points of m then S contains every point of m.

Let $\mathcal{S}$ be the set of subspaces of a generalized projective geometry ρ. $\mathcal{S}$, together with the partial ordering of set inclusion, is a lattice; this lattice has the property that for all subspaces S and T such that $S \subseteq T$, if there exists a maximal chain of finite length from S to T, then all maximal chains from S to T have the same finite length. If ρ is a projective geometry then all lines of ρ have the same cardinality of points; the *order* of ρ is the cardinal number n such that each line has $n+1$ points. If ρ is a projective geometry of dimension at least 3 (i.e., properly containing a plane) then the lattice of subspaces of ρ is the lattice of subspaces of a vector space over some skew field, and the order of the skew field equals the order of ρ [1, 4].

Theorem 3. *Let π be a connected semilinear incidence structure satisfying* (h1), (h2), (h4), *and*

(h3) *for any point x and line m not containing x, $t(x, m) \neq 1$.*

Then for some generalized projective geometry ρ and some $i \in \mathbb{Z}$, π is isomorphic to the incidence structure having $(i-1)$-dimensional subspaces of ρ as points, i-dimensional subspaces of ρ as lines, and set inclusion as the incidence relation.

Theorem 1 has the equivalent formulation that if π is a connected semilinear incidence structure satisfying (h1), (h2), (h3′), (h4), and $|P|>1$, $|L|>1$, then the generalized projective geometry ρ in the conclusion of Theorem 3 is a projective geometry of finite order.

3. Planes

In this section, starting with an incidence structure (P, L, I) satisfying Pasch's Axiom, we define a set E of planes, and show that the incidence structure $(L, E, \in)$ has many of the same properties as (P, L, I).

Lemma 3.1. *Let $\pi = (P, L, I)$ be a connected semilinear incidence structure satisfying* (h3). *and such that $|P|>1$, $|L|>1$. Then every point is on at least two lines and every line has at least two points.*

Proof. Let $x \in P$. If x is on all lines then x is at least two lines. If x is not on all lines then there is a line m not containing x such that $t(x, m) \neq 0$; since

$t(x, m) \neq 1$, then x is on at least two lines which intersect m. That every line has at least two points is shown by a dual argument.

Corollary. *Any two intersecting lines of π have a transversal.*

Through the remainder of this section we add (h1) to the hypotheses of Lemma 3.1: $\pi = (P, L, I)$ will be a connected semilinear incidence structure satisfying (h1) and (h3), and such that $|P| > 1$, $|L| > 1$.

For any pair of interesting lines m and n we let $e(m, n)$, called the *plane* determined by m and n, be the set of lines which either (i) are transversals of m and n, or (ii) contain $m \cap n$ and intersect at least one transversal of m and n. Since m and n satisfy condition (ii), $m, n \in e(m, n)$. Clearly, not all lines of $e(m, n)$ are concurrent. We let E denote the set of planes.

A *clique of lines* is a set of intersecting lines. Equivalently, a clique of lines is a clique in the line graph of π.

Lemma 3.2. *Every plane is a maximal clique of lines.*

Proof. Let lines m and n intersect at x. Let $h_1, h_2 \in e(m, n)$. To show h_1 intersects h_2 we need only consider the case where h_1 is a transversal of m and n, and h_2 contains x. Let $h_1 \cap m = y$ and $h_1 \cap n = z$. Now h_2 intersects some transversal t of m and n. Now t does not contain both y and z; say y is not on t. Then h_1 and h_2 are transversals of m and t, so they intersect. That $e(m, n)$ is not properly contained in any clique of lines is obvious.

Lemma 3.3. *Let K be a maximal clique of lines, and $m, n \in K (m \neq n)$. Then either $K = e(m, n)$ or all lines of K contain $m \cap n$.*

Proof. We will assume not all lines of K contain $m \cap n = x$ and show $K = e(m, n)$. Assume $t \in K$ and t does not contain x. Then t is a transversal of m and n. For any line $h \in K - \{m, n, t\}$, h intersects m, n, t. If h does not contain x then h is a transversal of m and n so $h \in e(m, n)$; if h contains x then $h \in e(m, n)$ since h intersects a transversal of m and n. Therefore $K \subseteq e(m, n)$. Since K is maximal, $K = e(m, n)$.

Corollary. *If m and n are distinct lines, e a plane, and $m, n \in e$, then $e = e(m, n)$.*

Note that the above lemma is equivalent to the assertion that $(L, E, \in)$ is semilinear.

A plane e will be said to *contain* a point x if for some line m containing x, $m \in e$.

Lemma 3.4. *Let m be a line, e a plane, and e contain at least two points of m. Then $m \in e$.*

Proof. Let x and y be points of m which are contained in e. Then there exist lines n_1, n_2 of e which contain x, y respectively. Since n_1 and n_2 are lines of the same plane, n_1 intersects n_2. Now m is a transversal of n_1 and n_2, so $m \in e(n_1, n_2) = e$.

Lemma 3.5. *If the plane e contains the point x then x is incident with at least two lines of e.*

Lemma 3.5 follows from Lemma 3.1, when e is viewed as an incidence structure. The next lemma merely states that two lines are on some common plane iff they are intersecting lines.

Lemma 3.6. *The adjacency graph of $(L, E, \in)$ is also the line graph of (P, L, I).*

Lemma 3.7. *$(E, L, \ni)$ satisfies Pasch's Axiom.*

Proof. Let lines m, n have a plane in common (equivalently, let m and n intersect at point x). Let lines h_1, h_2 intersect m and n but not belong to $e(m, n)$. To show that $(E, L, \ni)$ satisfies Pasch's Axiom we will show that h_1 and h_2 have a plane in common. This is equivalent to showing that h_1 and h_2 intersect. Since h_1 intersects m and n and is not in $e(m, n)$, then x is a point of h_1. Similarly x is a point of h_2. Therefore h_1 and h_2 intersect at x.

Lemma 3.8. *If the dual of π satisfies Pasch's Axiom then $(L, E, \in)$ satisfies Pasch's Axiom.*

Proof. In this proof if planes e_i and e_j have a line in common we call this line m_{ij}. Let e_1, e_2 be distinct planes and m_{12} exist. We are to show that for any planes e_3 *and* e_4 which are "transversals" of e_1 and e_2 (that is, planes e_3 and e_4 so that $m_{13}, m_{14}, m_{23}, m_{24}$ exist and are distinct from m_{12}), e_3 and e_4 have a line in common. Let $y_3 = m_{12} \cap m_{13} \cap m_{23}$ and $y_4 = m_{12} \cap m_{14} \cap m_{24}$. We distinguish two cases, according as y_3 and y_4 are equal or not.

Case 1. $y_3 \neq y_4$. Since m_{13} and m_{14} are distinct lines of e_1, they intersect at a point x_1. Similarly m_{23} and m_{24} intersect at a point x_2. Since x_1 is on e_1 but not on m_{12}, x_1 is not on e_2. Therefore $x_1 \neq x_2$. Since the dual of π satisfies Pasch's Axiom, the set of points adjacent to y_3 and y_4 but not on m_{12} is a clique; in particular x_1 is adjacent to x_2.

Let n be the line containing x_1 and x_2. Then n is a line of both e_3 and e_4.

Case 2. $y_3 = y_4$. Let u be a point of m_{12}, $u \neq y_3$. Let h_i be a line of e_i $(i = 1, 2)$ containing u and distinct from m_{12}. Let $e_5 = e(h_1, h_2)$. Now e_5 is a transversal of e_1 and e_2. Planes e_3 and e_5 are transversals of e_1 and e_2; by Case 1, m_{35} exists. Similarly m_{45} exists. Now, e_3 and e_4 are transversals of e_2 and e_5. If $m_{23} \neq m_{24}$, then e_3 and e_4 have a line in common by Case 1. If $m_{23} = m_{24}$, then e_3 and e_4 have the line m_{23} in common.

Lemma 3.9. *For any incidence structure $\rho = (X, M, J)$ let $T(\rho) = \{i \in \mathbb{Z}^+ : t(x, m) = i$ for some non-incident $x \in X$, $m \in M\}$. Then $T(L, E, \in) \subseteq T(\pi)$.*

Proof. Let $e \in E$, $m \in L$, and $m \notin e$. Let $\Delta(m)$ be the set of lines intersecting m; then $t(m, e) = |\Delta(m) \cap e|$. Let $t(m, e) > 0$. Let $n \in \Delta(m) \cap e$, and x be the point of intersection of m and n. Then $\Delta(m) \cap e$ is the set of lines of e containing x. Let h be a line of e not containing x. Then $\Delta(m) \cap e$ is the set of lines containing x and intersecting h. Therefore $t(m, e) = |\Delta(m) \cap e| = t(x, h) \in T(\pi)$.

Corollary. *$(L, E, \in)$ satisfies (h3). If π satisfies condition (h3') or (h3'') then $(L, E, \in)$ satisfies the same condition.*

Lemma 3.10. *If $r(\pi)$ is finite then $r((L, E, \in)) < r(\pi)$.*

Proof. Let $x \in P$ so that x is on exactly $r = r(\pi)$ lines. Let $m_1, m_2, \ldots, m_r$ be the lines containing x. Every plane containing m_1 contains m_i for some $i > 1$. Then $\{e(m_1, m_i): i > 1\}$ is the set of planes containing m_1. Since m_1 is on at most $r-1$ planes, $r((L, E, \in)) \leq r-1$.

Lemma 3.11. *One of the following occurs.*

(i) *$|E| = 1$, and all lines intersect.*

(ii) *Every line is one more than one plane, every pair of intersecting lines is contained in exactly two maximal cliques of lines, and not all lines intersect.*

Proof. By Lemma 3.1 either (i) $|E| = 1$ (so all lines intersect) or (ii) every line is on at least two planes. Assume (ii) holds. Let m and n be intersecting lines and $x = m \cap n$. Let K_x be the set of lines containing x. To show that m and n are contained in exactly two maximal cliques (namely K_x and $e(m, n)$) it is sufficient to show that $K_x \nsubseteq e(m, n)$. Let e' be a plane containing m and distinct from $e(m, n)$. Let h be a line of e' containing x and distinct from m. Then $h \in K_x - e(m, n)$, so $K_x \nsubseteq e(m, n)$. Clearly h does not intersect all lines of $e(m, n)$, so not all lines intersect.

We sum up the results of this section in Lemma 3.12.

Lemma 3.12. *If $\pi = (P, L, I)$ is a connected semilinear incidence structure satisfying (h1)–(h3), with $|P| > 1$, $|L| > 1$, and such that $r(\pi)$ is finite, then $(L, E, \in)$ is a connected semilinear incidence structure satisfying (h1)–(h3), and $r((L, E, \in)) < r(\pi)$.*

4. Proof of theorems

We state, and then prove, Theorems 1–3 in dualized form.

Theorems 1–3 (dualized). *Let* $\pi=(P, L, I)$ *be a connected semilinear incidence structure satisfying* (h1)–(h3), *and*

(h4*) *some point is on finitely many lines.*

Then for some generalized projective geometry ρ, π *is isomorphic to the incidence structure having i-dimensional and* $(i-1)$*-dimensional subspaces of* ρ *respectively as points and lines (and set inclusion as the incidence relation). If* $|P|>1$, $|L|>1$, *and* π *also satisfies* (h3′) *then* ρ *is a projective geometry. If* π *also satisfies* (h3″) *then for some set* X, $\rho\cong(X, \mathcal{P}_2(X), \subset)$.

Proof. We may assume that $|P|>1$ and $|L|>1$. We may also assume that the conclusion of this theorem is valid for all connected semilinear incidence structures ω satisfying (h1)–(h3), (h4*) for which $r(\omega)<r(\pi)$.

We let E be the set of planes, as in Section 3. Let $\pi'=(L, E, \in)$. Now π' is a connected semilinear incidence structure for which (h1)–(h3), (h4*) are valid, and $r(\pi')<r(\pi)$.

Case 1. $|E|=1$. Then all lines intersect. It is clear that the dual π^* of π is a generalized projective geometry. If π satisfies (h3′) then all members of P are incident with at least three members of L, so π^* is a projective geometry of finite order. If π satisfies (h3″) then all members of P are incident with exactly two members of L, so $\pi\cong(\mathcal{P}_2(L), L, \ni)$.

Case 2. $|E|>1$. Now π' satisfies (h1)–(h3), (h4*), and $r(\pi')<r(\pi)$. Then for some generalized projective geometry ρ, π' is isomorphic to the incidence structure having (i − 1)-dimensional and $(i-2)$-dimensional subspaces of ρ respectively as points and lines. (Also, if π satisfies (h3′) then so does π', so ρ is a projective geometry of finite order, and if π satisfies (h3″) then so does π', so $\rho\cong(X, \mathcal{P}_2(X), \in)$ for some set X.)

For all integers $j\geq 0$ let $\mathcal{S}_j$ be the set of j-dimensional subspaces of ρ. Let $\rho_{i-1}=(\mathcal{S}_{i-1}, \mathcal{S}_{i-2}, \supseteq)$ and $\rho_i=(\mathcal{S}_i, \mathcal{S}_{i-1}, \supseteq)$. Since $\pi'\cong\rho_{i-1}$ then there exist bijections $\sigma: L\to\mathcal{S}_{i-1}$ and $\tau: E\to\mathcal{S}_{i-2}$ such that for all $m\in L, e\in E, m^\sigma\supseteq e^\tau$ iff $m\in e$.

Let H be the adjacency graph of π' and $\bar{H}$ be the adjacency graph of ρ_{i-1}. Let C be the set of maximal cliques of H and $\bar{C}$ the set of maximal cliques of $\bar{H}$.

By Lemma 3.6 H is also the line graph of π. Therefore lines m and n are adjacent vertices of H if and only if they are intersecting lines. By Lemma 3.11 every pair of intersecting lines m and n is contained in exactly two maximal cliques: the set of lines in $e(n, n)$ and the set of lines containing $m\cap n$. There is an obvious bijection between $P\cup E$ and C (which maps each point to the set of lines containing it, and maps each plane to the set of lines of that plane). Let $\alpha: P\cup E\to C$ be this bijection.

$\bar{H}$ is the graph having $\mathcal{S}_{i-1}$ as vertex set, and two vertices adjacent if their meet is in $\mathcal{S}_{i-2}$. Every maximal clique in $\bar{H}$ is of one of two types: (1) the set of $(i-1)$-dimensional spaces containing an $(i-2)$-dimensional space, and (2) the set of $(i-1)$-dimensional spaces contained in an (i)-dimensional space. There is an obvious bijection from $\mathscr{S}_i \cup \mathscr{S}_{i-2}$ to $\bar{C}$. Let $\beta : \mathscr{S}_i \cup \mathscr{S}_{i-2} \to \bar{C}$ be this bijection.

Now σ and τ act as an isomorphism from π' to ρ_{i-1}, so σ is an isomorphism of the adjacency graphs of π' and ρ_{i-1}, that is, σ is an isomorphism from H to $\bar{H}$. Then σ acts as a bijection from C to $\bar{C}$. For each $e \in E$, σ maps the lines of e to the $(i-1)$-dimensional spaces which contain the $(i-2)$-dimensional space e^{τ}. Clearly σ gives a bjection between the maximal clique of H corresponding to planes and the maximal cliques of $\bar{H}$ corresponding to elements of $\mathscr{S}_{i-2}$. (In other words, σ gives a bijection between E^{α} and $\mathscr{S}^{\beta}_{i-2}$.) Therefore σ gives a bijection between P^{α} and $\mathscr{S}^{\beta}_i$. Then $\mu = \alpha\sigma\beta^{-1}$ is a bijection from P to $\mathscr{S}_i$, and σ and μ acts an an isomorphism from π to ρ_i.

5. Minimality of the hypotheses

We show that (h1)–(h4) is a minimal set of hypotheses for Theorem 3. To do this, for each hypothesis (hi) we will exhibit a connected semilinear incidence structure which satisfies all hypotheses except (hi). A generalized quadrangle satisfies all hypotheses except (h3).

Let ρ be a Steiner triple system on $v > 7$ points. It is clear that ρ satisfies all hypotheses except perhaps (h1). If ρ satisfies (h1), then ρ is a projective geometry of order 2, so $v+1$ is a power of 2. Hence if $v+1$ is not a power of 2 then ρ does not satisfy (h1).

Let ρ be a Steiner triple system on $v > 7$ points, where $v-1$ is not a power of 2. Then ρ^*, the dual of ρ, satisfies all hypotheses of Theorem 3 except (h2).

Let V be an infinite dimensional vector space, and $\mathscr{L}$ the lattice of subspaces of V. Let G be the graph having subspaces of V as vertices, and two subspaces x and y adjacent if x covers y or y covers x in $\mathscr{L}$. The set S_0 of all finite dimensional subspaces of V is a component of G. Let S be a component of G different from S_0. For any $x, y \in S$, $x \vee y$ may be spanned by $x \wedge y$ together with finitely many vectors of $x \vee y$. Hence the factor space $(x \vee y)/x$ is a finite dimensional vector space. Let $x_0 \in S$. For $y \in S$ we define $\operatorname{rank}(y) = \dim(x_0 \vee y)/x_0 - \dim(x_0 \vee y)/y$. Then rank is a dimension function of S as a sublattice of $\mathscr{L}$: if $y < z$ then $\operatorname{rank}(z) = \operatorname{rank}(y) + 1$ if and only if z covers y. Let $W_i = \{y \in S : \operatorname{rank}(y) = i\}$ for all integers i. Then $(W_{i+1}, W_i, \supseteq)$ is connected and semilinear, and satisfies all hypotheses of Theorem 1 except (h4).

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